Finite Element Solution of Longitudinally Magnetized Elliptical Gyromagnetic Waveguides

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Abstract—**One of the most common methods employed in the study of waveguides with irregular electric or magnetic walls is the finite element approach. This method is based on the minimization of a functional, the solution of which satisfies the boundary value problem. It is utilized in this paper to study the elliptical gyromagnetic waveguide with either an electric or a magnetic wall. The functionals for the four possible planar solutions are separately summarized.**

I. Introduction

THE MODERN approach to the derivation of propagation in inhomogeneous waveguides or in waveguides with irregular cross sections is often based on the derivation of a functional and the use of the finite element method [1], [2]. The functional encountered in gyromagnetic waveguides with the direct field along the direction of propagation has been developed in terms of the coupled longitudinal electric (E_z) and magnetic (H_z) fields entering into the description of the coupled wave equations of this type of problem [3]. Since the transverse fields may be specified from a knowledge of the longitudinal ones, this approach gives the complete solution to the problem. This is in fact a common procedure in the study of waveguide problems which support hybrid modes [4]–[8]. The functional in the situation for which the direction of the direct field makes an arbitrary angle with that of propagation has been historically posed for the gyrotropic situation using a three-component electric or magnetic field vector and in terms of the transverse electric and magnetic fields [9], [10]. If the direct field is parallel to that of the direction of propagation, then E_z and H_z are coupled. If the direct field and the direction of propagation are perpendicular to each other, then E_z and H_z are decoupled provided the alternating fields do not vary along the direction of the direct field. The functional for the first arrangement has been tackled using the finite difference method [3] and that for the second one using the finite element method [10], [11]. The purpose of this paper is to investigate propagation along an elliptical gyromagnetic waveguide with electric or magnetic walls with the direct field parallel to that of propagation using the finite element method. Fig. 1 depicts the case of a longitudinally magnetized elliptical waveguide with an electric wall.

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The required functional is a scalar quantity which may be constructed by starting with a Rayleigh quotient [3], [12] and expressing it as a quadratic form [4], [5], [13] or by constructing the quadratic form directly from a knowledge of the coupled or uncoupled wave equations [9], [14]–[16]. It is reproduced here by directly constructing it in terms of the properties of a quadratic form as a preamble to carrying out some calculations. If the minimum of the functional corresponds to the solution of the boundary value problem then the final result is the energy functional of the system. In such cases the quadratic form is understood to be an energy function. While this approach gives the classical result [3], the language is more akin to that met in network rather than in electromagnetic theory [17]. Therefore it may be more comprehensible to the nonspecialist engineer. The solutions also correctly reduce to those met frequently in the description of the planar gyromagnetic problems with electric and magnetic walls in all combinations [18]–[20]. The cutoff space for this type of waveguide has been dealt with separately in [20] and [21]. The finite element method is currently being employed to analyze lossy three-dimensional problems through a variational approach [22]–[24] and by using alternative weighted residual schemes [25], [26].

II. Derivation of a Functional

The solution to irregular isotropic or gyromagnetic waveguides is often solved by constructing a quadratic form from a knowledge of the wave equation in either the z-directed scalar [3]–[8] or the three-dimensional vector [11]–[13], [16] form. It may be readily demonstrated that all energy functions have the nature of a quadratic form [17]. A property of any waveguide is that its field distribution in the transverse plane may always be formed from a knowledge of E_z and H_z. In a gyromagnetic waveguide magnetized along the direction of propagation, E_z and H_z are coupled. The classic derivation of the required functional starts with the coupled wave equations

$$L_{ee}E_z + L_{eh}H_z = 0 \tag{1a}$$

$$L_{he}E_z + L_{hh}H_z = 0. \tag{1b}$$

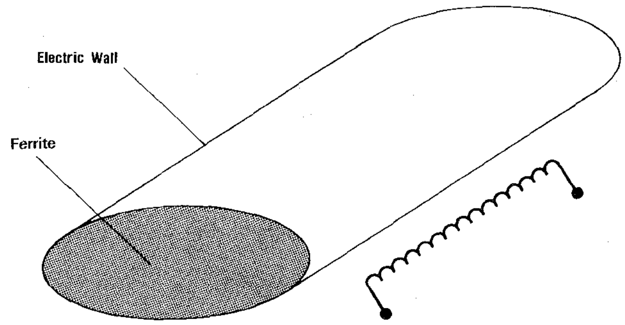

Fig. 1. Schematic diagram of elliptical gyromagnetic waveguide with an electric wall.

L_{ee}, L_{eh}, L_{he}, and L_{hh} are dependent upon the propagation constant, the frequency, the constitutive parameters, and second-order linear derivatives. This system of equations may be written in matrix form as

$$\begin{bmatrix} L_{ee} & L_{eh} \\ L_{he} & L_{hh} \end{bmatrix}\begin{bmatrix} E_z \\ H_z \end{bmatrix} = [0]. \quad (2)$$

The square matrix in this relationship is known as the operator and is denoted by

$$\bar{L} = \begin{bmatrix} L_{ee} & L_{eh} \\ L_{he} & L_{hh} \end{bmatrix}. \quad (3)$$

An energy function can be associated with the square matrix $\bar{L}$ and column vector

$$\begin{bmatrix} E_z \\ H_z \end{bmatrix}$$

by forming the related quadratic form

$$[E_z^* \quad H_z^*]\begin{bmatrix} L_{ee} & L_{eh} \\ L_{he} & L_{hh} \end{bmatrix}\begin{bmatrix} E_z \\ H_z \end{bmatrix}$$

and by recognizing that all energy functions have this type of form. If the quadratic form is greater than but not equal to zero it is said to be positive definite (P.D.). If it is equal to or greater than zero it is said to be positive semidefinite (P.S.D.). In the case of a lossless homogeneous gyromagnetic waveguide the tensor permeability is Hermitian, and a property of the matrix $\bar{L}$ of the quadratic form is that it is self-adjoint. A necessary condition for the resulting matrix eigenvalue problem to produce real eigenvalue frequencies is that the quadratic form be P.S.D. This condition requires that the elements of the matrix $\bar{L}$ must satisfy

$$L_{ee} \geqslant 0 \quad (4a)$$

$$L_{hh} \geqslant 0 \quad (4b)$$

$$(L_{ee}L_{hh} - L_{eh}L_{he}) \geqslant 0. \quad (4c)$$

An energy function of this type, once integrated over the cross-sectional area of the waveguide, is referred to as the functional of the system. This operation is usually written as

$$F(E_z, H_z) = \left\langle \begin{bmatrix} E_z \\ H_z \end{bmatrix}, \begin{bmatrix} L_{ee} & L_{eh} \\ L_{he} & L_{hh} \end{bmatrix}\begin{bmatrix} E_z \\ H_z \end{bmatrix} \right\rangle. \quad (5)$$

This is given in integral form by

$$F(E_z, H_z) = \iint_s E_z^* L_{ee} E_z\, ds + \iint_s E_z^* L_{eh} H_z\, ds + \iint_s H_z^* L_{he} E_z\, ds + \iint_s H_z^* L_{hh} H_z\, ds. \quad (6)$$

In order to obtain a complete solution of the boundary value problem it is necessary to minimize the quadratic functional in conjunction with the boundary conditions of the waveguide problem.

III. The Matrix of the Form in Gyromagnetic Waveguides

The derivation starts by decomposing the electric and magnetic fields ($\boldsymbol{E}$ and $\boldsymbol{H}$) into transverse ($\boldsymbol{E}_t, \boldsymbol{H}_t$) and longitudinal components ($\boldsymbol{a}_z E_z, \boldsymbol{a}_z H_z$). Adopting this notation and assuming the time and z variation to be

$$\exp(j\omega t)$$
$$\exp(-j\beta z)$$

gives

$$\boldsymbol{E} = (\boldsymbol{E}_t + \boldsymbol{a}_z E_z)\exp(-j\beta z)\exp(j\omega t) \quad (7)$$

$$\boldsymbol{H} = (\boldsymbol{H}_t + \boldsymbol{a}_z H_z)\exp(-j\beta z)\exp(j\omega t). \quad (8)$$

The constitutive parameters in a gyromagnetic waveguide with the direction of the direct magnetic field along that of propagation are given in terms of a tensor permeability ($[\mu]$) and a scalar permittivity (ϵ) by

$$[\mu] = \mu_0 \begin{bmatrix} \mu & -j\kappa & 0 \\ j\kappa & \mu & 0 \\ 0 & 0 & \mu_z \end{bmatrix} \quad (9)$$

$$\epsilon = \epsilon_0 \epsilon_f \quad (10)$$

respectively, where ϵ_0 and μ_0 denote the permittivity and permeability of free space.

The wave equations for E_z and H_z are in this instance coupled. The solution to this problem is a standard result in the literature [27]. This gives the entries of the matrix of the form as

$$L_{ee} = \nabla_t^2 - \beta^2 + k_0^2 \mu_{\text{eff}} \epsilon_f \quad (11a)$$

$$L_{eh} = j\sqrt{\left(\frac{\mu_0}{\epsilon_0}\right)} \frac{\beta k_0 \kappa \mu_z}{\mu} \quad (11b)$$

$$L_{he} = L_{eh}^* \quad (11c)$$

$$L_{hh} = \frac{\mu_0 \mu_z}{\epsilon_0 \epsilon_f}\left(\nabla_t^2 + \frac{k_1^2 \mu_z}{\mu}\right) \quad (11d)$$

where

$$k_1^2 = -\beta^2 + k_0^2 \mu \epsilon_f \quad (12)$$

and

$$\mu_{\text{eff}} = \frac{\mu^2 - \kappa^2}{\mu} \tag{13a}$$

$$k_0 = \frac{2\pi}{\lambda_0}. \tag{13b}$$

In this description of the matrix of the form each element is a function of both β and k_0 to first and second order. It is usual to recast it, for computational purposes, in terms of the normalized propagation constant ($\bar{\beta} = \beta/k_0$) and k_0^2 [3]. Using (11c) and (11d) the coupled wave equation in (1b) may be written as

$$H_z = j\sqrt{\left(\frac{\epsilon_0}{\mu_0}\right)}\frac{\beta\kappa k_0\epsilon_f}{\mu_z k_1^2}E_z - \frac{\mu}{\mu_z k_1^2}\nabla_t^2 H_z \tag{14}$$

and substituting it together with (11a) and (11b) into the other coupled wave equation defined by (1a) gives

$$L'_{ee} = a\nabla_t^2 + k_0^2\sqrt{\left(\frac{\epsilon_0}{\mu_0}\right)}\epsilon_f \tag{15a}$$

$$L'_{eh} = -jb\nabla_t^2. \tag{15b}$$

Starting with the coupled wave equation (1a) and making use of the relationships in (11a) and (11b) gives

$$L'_{he} = L'^{*}_{eh} \tag{15c}$$

$$L'_{hh} = c\nabla_t^2 + k_0^2\sqrt{\left(\frac{\mu_0}{\epsilon_0}\right)}\mu_z. \tag{15d}$$

E_z and H_z are coupled through the imaginary term L'_{eh}, which is nonzero for β and κ nonzero. The quantities a, b, c, d, e, and f are given in the Appendix.

IV. Quadratic Functional for a Gyromagnetic Region

The quadratic functional applicable to a finite space region is readily deduced by combining (6) and (15). The result is in terms of second-order linear derivatives. Integrating this equation by parts and applying Green's identity in a plane reduces it to one involving first-order linear derivatives only. The products appearing in the quadratic form then become

$$\iint_s E_z^* L'_{ee} E_z\, ds = -a\iint_s |\nabla_t E_z|^2\, ds + a\int_\xi E_z^* \frac{\partial E_z}{\partial n}\, dl + k_0^2\sqrt{\left(\frac{\epsilon_0}{\mu_0}\right)}\epsilon_f \iint_s |E_z|^2\, ds \tag{16a}$$

$$\iint_s E_z^* L'_{eh} H_z\, ds = jb\iint_s \nabla_t E_z^* \cdot \nabla_t H_z\, ds - jb\int_\xi E_z^* \frac{\partial H_z}{\partial n}\, dl \tag{16b}$$

$$\iint_s H_z^* L'_{he} E_z\, ds = -jb\iint_s \nabla_t H_z^* \cdot \nabla_t E_z\, ds + jb\int_\xi H_z^* \frac{\partial E_z}{\partial n}\, dl \tag{16c}$$

$$\iint_s H_z^* L'_{hh} H_z\, ds = -c\iint_s |\nabla_t H_z|^2\, ds + c\int_\xi H_z^* \frac{\partial H_z}{\partial n}\, dl + k_0^2\sqrt{\left(\frac{\mu_0}{\epsilon_0}\right)}\mu_z \iint_s |H_z|^2\, ds. \tag{16d}$$

In this derivation s is a closed surface, and the boundary of s is a closed curve ξ. The positive side of ξ is defined to be that in which an observer would travel to have the interior of s on the left side. $\bar{\xi}$ is the unit vector tangent to ξ in the positive direction. $\boldsymbol{n}$ is the outward unit vector normal to s.

V. Elliptical Waveguide with an Electric Wall

The boundary conditions in the electric wall problem

$$E_\xi = 0 \tag{17a}$$

$$E_z = 0 \tag{17b}$$

are applied to the tangential electric field E_ξ:

$$E_\xi = -jf\frac{\partial E_z}{\partial \xi} + b\frac{\partial E_z}{\partial n} - d\frac{\partial H_z}{\partial \xi} - jc\frac{\partial H_z}{\partial n} \tag{17c}$$

to give the required boundary equation as [28]

$$b\frac{\partial E_z}{\partial n} - d\frac{\partial H_z}{\partial \xi} - jc\frac{\partial H_z}{\partial n} = 0. \tag{17d}$$

Introducing this boundary equation into the functional for the finite gyromagnetic region and making use of the following relationship between the coupled variables,

$$j\iint_s \nabla_t E_z^* \cdot \nabla_t H_z - (\nabla_t E_z^* \cdot \nabla_t H_z)^*\, ds = -2\iint_s \nabla_t E_z \cdot \nabla_t H_z\, ds \tag{18}$$

gives the required form of the functional for a longitudinally gyromagnetic waveguide with an electric wall:

$$F(E_z, H_z) = -a\iint_s |\nabla_t E_z|^2\, ds + k_0^2\sqrt{\left(\frac{\epsilon_0}{\mu_0}\right)}\epsilon_f \iint_s |E_z|^2\, ds - 2b\iint_s \nabla_t E_z \cdot \nabla_t H_z\, ds - c\iint_s |\nabla_t H_z|^2\, ds + jd\int_\xi H_z^* \frac{\partial H_z}{\partial \xi}\, dl + k_0^2\sqrt{\left(\frac{\mu_0}{\epsilon_0}\right)}\mu_z \iint_s |H_z|^2\, ds. \tag{19}$$

The electromagnetic field solution may be obtained from the stationary condition of the functional. Adopting the Rayleigh–Ritz minimization method yields the following matrix eigenvalue equation:

$$\begin{bmatrix} a[S] & b[S] \\ b[S] & c[S] \end{bmatrix}\begin{bmatrix} e_z \\ h_z \end{bmatrix} - jd\begin{bmatrix} [0] & [0] \\ [0] & [C] \end{bmatrix}\begin{bmatrix} e_z \\ h_z \end{bmatrix} = k_0^2 \begin{bmatrix} \sqrt{\left(\frac{\epsilon_0}{\mu_0}\right)}\epsilon_f[T] & [0] \\ [0] & \sqrt{\left(\frac{\mu_0}{\epsilon_0}\right)}\mu_z[T] \end{bmatrix}\begin{bmatrix} e_z \\ h_z \end{bmatrix} \tag{20}$$

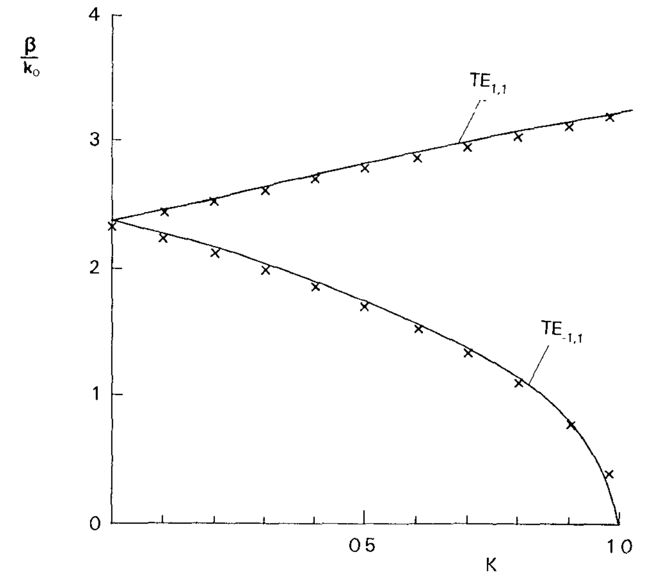

Fig. 2. Comparison between exact and finite element method of the propagation constants of the $TE_{\pm 1\,1}$ mode in a gyromagnetic circular waveguide with an electric wall ($k_0R = 0.6$, $\mu = 1$, $\mu_z = 1$, $\epsilon_f = 15$).

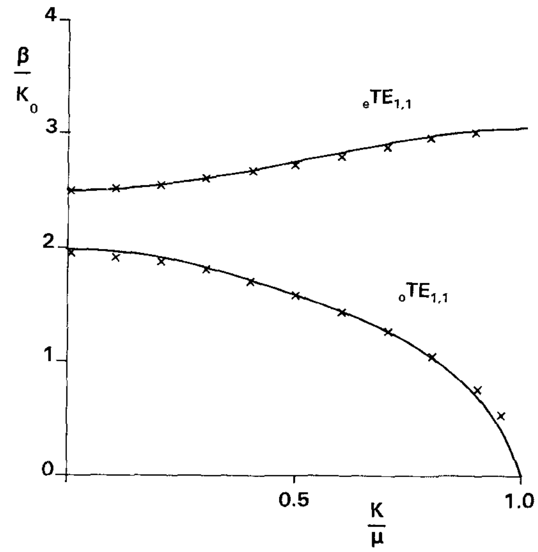

Fig. 3. Comparison between exact and finite element method of the propagation constants of the $_eTE_{1,1}$ and $_oTE_{1,1}$ modes in an elliptical gyromagnetic waveguide with an electric wall ($k_0R = 0.9696$, $\mu = 1$, $\mu_z = 1$, $\epsilon_f = 10$, $e = 0.648$).

for a waveguide of general cross section bounded by an electric wall. This quantity differs from the isotropic formulation in the coupling terms [4]–[8]. It is of note that the imaginary skew-symmetric part of the Hermitian matrix only makes a contribution to the real part of the Hermitian form, and the resultant form is therefore real and will produce real eigenfrequencies. $[S]$ and $[T]$ are real, symmetric square matrices [2]:

$$S_{i,j} = \iint_s \nabla_t \alpha_i \cdot \nabla_t \alpha_j \, ds \tag{21a}$$

$$T_{i,j} = \iint_s \alpha_i \alpha_j \, ds \tag{21b}$$

and the imaginary skew-symmetric square matrix $[C]$ is given by [18]

$$C_{i,j} = \int_\xi \alpha_i \frac{\partial \alpha_j}{\partial \xi} \, dl. \tag{21c}$$

Here $\alpha_{i,j}$ are a set of linearly independent real basis functions. The constitutive parameters and $\bar{\beta}$ are inputs to the finite element program; k_0^2 is the calculated eigenvalue. From a knowledge of the relationship between $\bar{\beta}$ and k_0 over a range of κ values, it is possible to plot $\bar{\beta}$ versus κ diagrams.

Fig. 2 illustrates the agreement between the exact and finite element solutions in the case of a round waveguide. Fig. 3 depicts the phase constants for an elliptical gyromagnetic waveguide. The segmentation employed here consists of 15 third-order elements. This is indicated in Fig. 4.

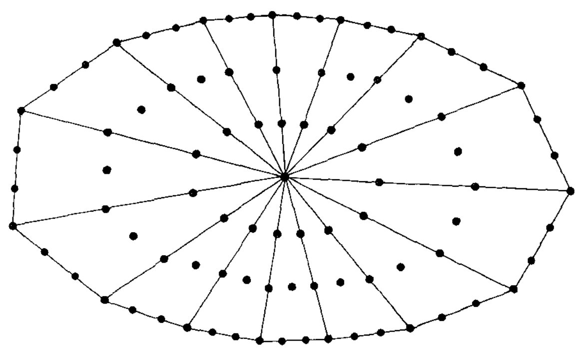

Fig. 4. Segmentation of an elliptical cross section into 15 third-order triangles.

VI. Elliptical Waveguide with Magnetic Walls

The boundary conditions relating to a waveguide with a magnetic wall surface,

$$H_\xi = 0 \tag{22a}$$

$$H_z = 0 \tag{22b}$$

reduce the expression for the tangential magnetic field,

$$H_\xi = e\frac{\partial E_z}{\partial \xi} + ja\frac{\partial E_z}{\partial n} - jf\frac{\partial H_z}{\partial \xi} + b\frac{\partial H_z}{\partial n} \tag{22c}$$

to the following boundary equation:

$$b\frac{\partial H_z}{\partial n} + e\frac{\partial E_z}{\partial \xi} + ja\frac{\partial E_z}{\partial n} = 0. \tag{22d}$$

Once again making use of the relationship between the coupled variables (18), the functional for the magnetic wall waveguide is obtained as

$$\begin{aligned} F(E_z, H_z) = &- a\iint_s |\nabla_t E_z|^2 \, ds + je\int_\xi E_z^* \frac{\partial E_z}{\partial \xi} \, dl \\ &+ k_0^2 \sqrt{\left(\frac{\epsilon_0}{\mu_0}\right)} \epsilon_f \iint_s |E_z|^2 \, ds \\ &- 2b\iint_s \nabla_t E_z \cdot \nabla_t H_z \, ds - c\iint_s |\nabla_t H_z|^2 \, ds \\ &+ k_0^2 \sqrt{\left(\frac{\mu_0}{\epsilon_0}\right)} \mu_z \iint_s |H_z|^2 \, ds. \end{aligned} \tag{23}$$

The Rayleigh–Ritz minimization of this functional gives

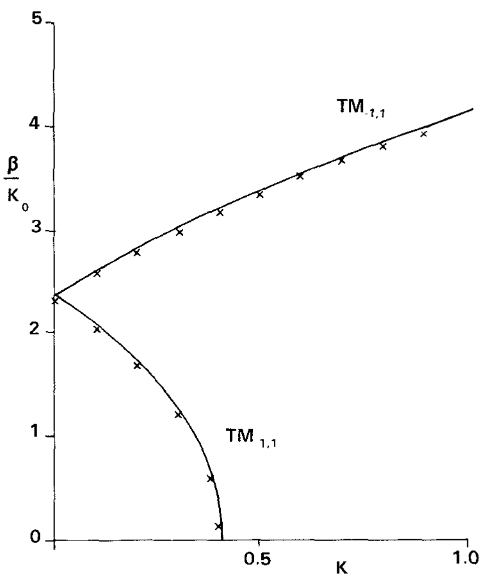

Fig. 5. Comparison between exact and finite element method of the propagation constants of the $TM_{\pm 1,1}$ mode in a circular gyromagnetic waveguide with a magnetic wall ($k_0 R = 0.6$, $\mu = 1$, $\mu_z = 1$, $\epsilon_f = 15$).

the matrix eigenvalue equation as

$$\begin{bmatrix} a[S] & b[S] \\ b[S] & c[S] \end{bmatrix}\begin{bmatrix} e_z \\ h_z \end{bmatrix} - je\begin{bmatrix} [C] & [0] \\ [0] & [0] \end{bmatrix}\begin{bmatrix} e_z \\ h_z \end{bmatrix} = k_0^2 \begin{bmatrix} \sqrt{\left(\frac{\epsilon_0}{\mu_0}\right)}\epsilon_f[T] & [0] \\ [0] & \sqrt{\left(\frac{\mu_0}{\epsilon_0}\right)}\mu_z[T] \end{bmatrix}\begin{bmatrix} e_z \\ h_z \end{bmatrix}. \quad (24)$$

Fig. 5 compares the finite element solution and the exact one for a circular gyromagnetic waveguide with a magnetic wall. Fig. 6 indicates one result in the case of an elliptical waveguide. This solution is compatible with the approximate phase constants deduced from the split frequencies of a planar circuit with magnetic sidewalls and top and bottom electric walls [20].

VII. Functionals for Planar Gyromagnetic Resonators

The cutoff plane of the gyromagnetic waveguide problem with an electric sidewall may be constructed by minimizing the functionals of the related planar geometries with either magnetic or electric top and bottom walls together with a knowledge of the rules governing mode nomenclature [29]–[31]. The functional for the transverse electric (TE) modes of a planar circuit with top and bottom magnetic walls is obtained by letting

$$\beta = 0 \quad (25a)$$

$$E_z = 0 \quad (25b)$$

in (19). This gives

$$F(H_z) = -\iint_s |\nabla_t H_z|^2\, ds + k_0^2 \epsilon_f \mu_z \iint_s |H_z|^2\, ds. \quad (26)$$

The functional associated with the transverse magnetic (TM) modes of the planar resonator bounded by an elec-

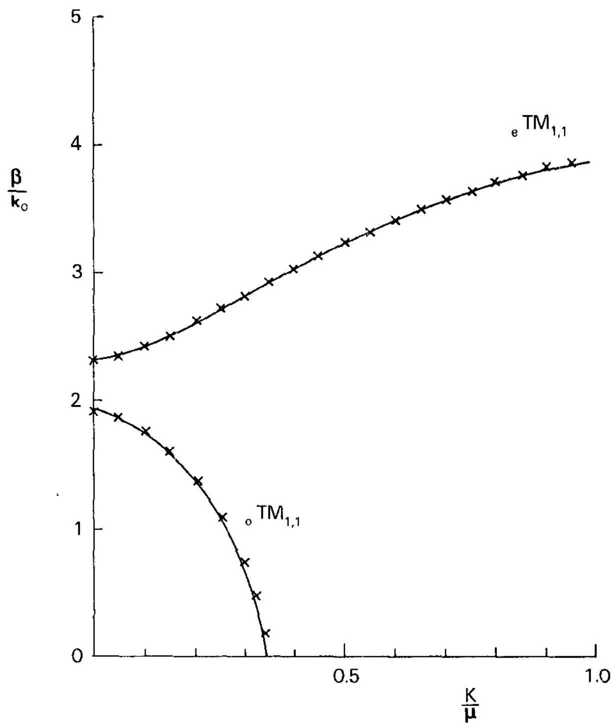

Fig. 6. Comparison between perturbation theory and the finite element method of the propagation constants of the $_eTM_{1,1}$ and $_oTM_{1,1}$ modes in an elliptical gyromagnetic waveguide with a magnetic wall ($k_0 R = 0.6$, $\mu = 1$, $\mu_z = 1$, $\epsilon_f = 15$, $e = 0.4$).

tric sidewall and top and bottom electric walls is derived by letting

$$\beta = 0 \quad (27a)$$

$$H_z = 0 \quad (27b)$$

in (19). The result is

$$F(E_z) = -\iint_s |\nabla_t E_z|^2\, ds + k_0^2 \epsilon_f \mu_{\text{eff}} \iint_s |E_z|^2\, ds. \quad (28)$$

These functionals, (26) and (28), have been used in [20] to construct the cutoff space of elliptical gyromagnetic waveguides with electric walls.

The functional for the planar circuit with a magnetic sidewall may also be obtained from the related functional for the magnetic wall waveguide. The conditions for electric top and bottom walls, stated in (27), reduce the functional in (23) to the following:

$$F(E_z) = -\iint_s |\nabla_t E_z|^2\, ds + j\frac{\kappa}{\mu}\int_\xi E_z^* \frac{\partial E_z}{\partial \xi}\, dl + k_0^2 \epsilon_f \mu_{\text{eff}} \iint_s |E_z|^2\, ds. \quad (29)$$

This is exactly the result given in [18] and [19] and may be used to deal with the planar gyromagnetic problem. The imaginary part of (29) results in split cutoff numbers for the TM modes of this waveguide.

Similarly the conditions for magnetic top and bottom walls, stated in (25), applied to (23) yield the functional for the planar geometry with magnetic side, and top and bottom walls as

$$F(H_z) = -\iint_s |\nabla_t H_z|^2\, ds + k_0^2 \epsilon_f \mu_z \iint_s |H_z|^2\, ds. \quad (30)$$

This functional is identical to that in (26) but the solution to this problem requires that the functional be minimized in conjunction with the boundary condition that H_z is zero on the magnetic sidewall.

The functionals given by (29) and (30) may be used to construct the TM and TE modes respectively in the cutoff plane of the related gyromagnetic waveguide problem with a magnetic wall [20].

VIII. Conclusion

The classic method employed to derive a functional is to construct a quadratic form, by premultiplying the wave equation by the conjugate field, and then to integrate this quantity over the cross section of the problem region. This procedure has been utilized in this paper to construct a solution of the longitudinally magnetized gyromagnetic problem using the z-directed coupled wave equations. It has been used in conjunction with the finite element method to evaluate the propagation constants of an elliptical gyromagnetic waveguide with either an electric or a magnetic wall. The well-known functionals associated with the four possible planar circuits made up of electric and magnetic side and top and bottom walls are directly obtained from those of the related waveguide problems.

Appendix

The coefficients met in the text are defined below:

$$a = \epsilon_f \sqrt{\left(\frac{\epsilon_0}{\mu_0}\right)} \frac{\left(\epsilon_f \mu - \bar{\beta}^2\right)}{\left[\bar{\beta}^2 - \epsilon_f(\mu+\kappa)\right]\left[\bar{\beta}^2 - \epsilon_f(\mu-\kappa)\right]}$$

$$b = \frac{\epsilon_f \kappa \bar{\beta}}{\left[\bar{\beta}^2 - \epsilon_f(\mu+\kappa)\right]\left[\bar{\beta}^2 - \epsilon_f(\mu-\kappa)\right]}$$

$$c = \mu \sqrt{\left(\frac{\mu_0}{\epsilon_0}\right)} \frac{\left(\epsilon_f \mu_{\text{eff}} - \bar{\beta}^2\right)}{\left[\bar{\beta}^2 - \epsilon_f(\mu+\kappa)\right]\left[\bar{\beta}^2 - \epsilon_f(\mu-\kappa)\right]}$$

$$d = \sqrt{\left(\frac{\mu_0}{\epsilon_0}\right)} \frac{\bar{\beta}^2 \kappa}{\left[\bar{\beta}^2 - \epsilon_f(\mu+\kappa)\right]\left[\bar{\beta}^2 - \epsilon_f(\mu-\kappa)\right]}$$

$$e = \sqrt{\left(\frac{\epsilon_0}{\mu_0}\right)} \frac{\epsilon_f^2 \kappa}{\left[\bar{\beta}^2 - \epsilon_f(\mu+\kappa)\right]\left[\bar{\beta}^2 - \epsilon_f(\mu-\kappa)\right]}$$

$$f = \frac{\bar{\beta}\left(\epsilon_f \mu - \bar{\beta}^2\right)}{\left[\bar{\beta}^2 - \epsilon_f(\mu+\kappa)\right]\left[\bar{\beta}^2 - \epsilon_f(\mu-\kappa)\right]}$$

$$\bar{\beta} = \frac{\beta}{k_0}.$$

References

[1] S. Ahmed, "Finite element method for waveguide problems," *Electron Lett.*, vol. 4, pp. 387–389, Sept. 1968.

[2] P. Silvester, "A general high order finite element waveguide analysis program," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 204–210, Apr. 1969.

[3] C. D. Hannaford, "Finite difference/variational method for inhomogeneous, isotropic or anisotropic waveguides," Ph.D. dissertation, University of Leeds, Leeds, England, 1967.

[4] S Ahmed and P Daly, "Finite-element methods for inhomogeneous waveguides," *Proc. Inst. Elec. Eng.*, vol. 116, pp. 1661–1664, Oct. 1969.

[5] P. Daly, "Finite-element coupling matrices," *Electron. Lett.*, vol. 5, pp. 613–615, Nov 1969.

[6] Z. J. Csendes and P. Silvester, "Numerical solution of dielectric loaded waveguides—Part 1: Finite element analysis," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 1124–1131, Dec. 1970.

[7] N. Mabaya, P. E. Lagasse, and P. Vandenbulcke, "Finite element analysis of optical waveguides," *IEEE Trans Microwave Theory Tech.*, vol. MTT-29, pp. 600–605, 1981.

[8] P. Daly, "Hybrid-mode analysis of microstrip by finite-element methods," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 19–25, Jan. 1971.

[9] A. Konrad, "Vector variational formulation of electromagnetic fields in anisotropic media," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-24, pp. 553–559, 1976.

[10] T. Angkaew, M. Matsuhan, N. Kumagai, "Finite element analysis of waveguide modes: A novel approach that eliminates spurious modes," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-35, pp. 117–123, Feb. 1987.

[11] A. Konrad, "High order triangular finite elements for electromagnetic waves in anisotropic media," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-25, pp. 353–360, May 1977.

[12] A. D. Berk, "Variational principles for electromagnetic resonators and waveguides," *IRE Trans. Antennas Propagat.*, vol. AP-4, pp. 104–111, Apr. 1956.

[13] B. M. A. Rahman and J. B. Davies, "Finite-element analysis of optical and microwave waveguide problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-32, pp. 20–28, 1984.

[14] I. Stakgold, *Boundary Value Problems of Mathematical Physics*, vols. I and II. New York: Macmillan, 1968.

[15] A. Wexler, "Computation of electromagnetic fields," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, Aug. 1969.

[16] W. J. English and F. J. Young, "An E vector variational formulation of Maxwell equations for cylindrical waveguide problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 40–46, Jan. 1971.

[17] N. Balabanian and T. A. Bickart, *Electrical Network Theory*. New York: Wiley, 1969.

[18] T. Miyoshi, S. Yamaguchi, and S. Goto, "Ferrite planar circuits in microwave integrated circuits," *IEEE Trans Microwave Theory Tech.*, vol. MTT-25, July 1977.

[19] R. W. Lyon and J. Helszajn, "A finite element analysis of planar circulators using arbitrarily shaped resonators," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-30, pp. 1967–1974, Nov. 1982.

[20] J. Helszajn and A. Gibson, "Cut-off spaces of elliptical gyromagnetic planar circuits and waveguides using finite elements," *IEEE Trans. Microwave Theory Tech.*, vol. 37, pp. 71–80, Jan. 1989.

[21] W. E. Salmond and C. Yeh, "Ferrite filled elliptical waveguides," *J. Appl. Phys.*, vol. 41, no. 8, pp. 3210–3226, July 1970.

[22] A. D. McAuley, "Variational finite element solution for dissipative waveguides and transportation systems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-25, pp. 382–392, May 1977.

[23] C. H. Chen and D. L Chuen, "The variational principle for non-self adjoint-problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-28, pp. 878–886, Aug. 1980.

[24] J P. Webb, G. L. Maile, R. L. Ferrari, "Finite element solution of three dimensional electromagnetic problems," *Proc. Inst. Elec. Eng.*, vol. 130, pt. H, no. 2, pp. 153–159, Mar. 1983.

[25] J. Simkin and C. W. Trowbridge, "Three-dimensional nonlinear electromagnetic field computations using scalar potentials," *Proc. Inst. Elec. Eng.*, vol. 127, pt. B, pp. 368–374, 1980.

[26] C. S. Biddlecombe, E. A Heighway, J. Simkin, and C W. Trowbridge, "Methods for eddy current computations in three dimensions," *IEEE Trans. Magn.*, vol. MAG-18, pp. 492–497, 1982.

[27] M. L. Kales, "Modes in waveguides containing ferrites," *J. Appl. Phys.*, vol. 24, pp. 604–608, 1953.

[28] L. G. Chambers, "Propagation in a ferrite filled waveguide," *J. Mech. Appl. Maths.*, pp 435–447, 1955

[29] R. A Waldron, "Properties of ferrite-loaded cylindrical waveguides in the neighbourhood of cut-off," *Proc. Inst. Elec. Eng.*, vol. 109B, suppl. 21, pp. 90–94, 1962.

[30] R. A. Waldron, *Theory of Waveguides and Cavities*. McLaren, 1967.
[31] J. Helszajn and A. Gibson, "Mode nomenclature of circular gyromagnetic and anisotropic waveguides with magnetic and open walls," *Proc. Inst. Elec. Eng.*, pt. H, Dec. 1987.

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